JOURNAL OF MULTIVARIATE ANALYSIS 10, 296–318 (1980)

Brunn–Minkowski Inequality and Its Aftermath

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Communicated by P. R. Krishnaiah

1. INTRODUCTION

The following inequality of Brunn–Minkowski for convex sets in R^n has led to many important results in statistical distribution theory and multivariate statistical inference.

THEOREM 1. *Let A_1 and A_2 be two non-empty convex sets in R^n. Then*

$$V_n(A_1 + A_2) \geqslant [\{V_n(A_1)\}^{1/n} + \{V_n(A_2)\}^{1/n}]^n, \tag{1.1}$$

where V_n stands for the n-dimensional volume, and $A_1 + A_2$ denotes the Minkowski sum of A_1 and A_2.

This inequality was first proved by Brunn [8] in 1887 and the conditions for equality to hold were derived by Minkowski [36] in 1919. Later, in 1935, Lusternik [34] generalized this result for non-empty arbitrary measurable sets A_1 and A_2 and derived conditions for equality to hold. Alternative and somewhat rigorous proof of Lusternik's result was given by Henstock and Macbeath [27] in 1953, and by Hadwiger and Ohman [24] in 1956–1959. Lusternik's conditions for equality were also corrected by Henstock and Macbeath [27].

First we shall consider the following generalization of Brunn–Minkowski–Lusternik inequality.

THEOREM 2. *Let f_0 and f_1 be two non-negative Borel-measurable functions on R^n with non-empty supports S_0 and S_1, respectively. Assume that f_0 and f_1 are integrable with respect to the Lebesgue measure μ_n on R_n.*

Received March 28, 1979; revised February 8, 1980.

Key Words: Brunn–Minkowski–Lusternik inequality; generalizations, s-unimodal functions.
AMS Classification: 26A86, 62H99.

* Partly supported by a grant from the Mathematics Division, U. S. Army Research Office, Durham, N. C. Grant DAAG-29-76-G-0038.

0047-259X/80/030296–23$02.00/0

Let $\theta (0 < \theta < 1)$ be a fixed number and f be a non-negative, measurable function on R^n such that

$$f(x) \geqslant M_\alpha [f_0(x_0), f_1(x_1); \theta], \tag{1.2}$$

whenever $x = (1-\theta)\, x_0 + x_1$ with $x_0 \in S_0$, $x_1 \in S_1$; $-1/n \leqslant \alpha \leqslant +\infty$. Then

$$\int_{(1-\theta)S_0 + \theta S_1} f(x)\, dx \geqslant M_{\alpha_n^*} \left[\int_{R^n} f_0(x)\, dx, \int_{R^n} f_1(x)\, dx; \theta \right], \tag{1.3}$$

where

$$\begin{aligned} \alpha_n^* &= \alpha/(1+n\alpha), && \textit{for} \quad -1/n < \alpha < +\infty, \\ &= 1/n, && \textit{for} \quad \alpha = +\infty, \\ &= -\infty, && \textit{for} \quad \alpha = -1/n. \end{aligned} \tag{1.4}$$

The generalized mean function M_α is defined as follows [26]. For non-negative a_0 and a_1

$$\begin{aligned} M_\alpha(a_0, a_1; \theta) &= [(1-\theta)\, a_0^\alpha + \theta a_1^\alpha]^{1/\alpha}, \\ &\qquad \text{if} \quad 0 < \alpha < \infty, \text{or if} -\infty < \alpha < 0 \text{ and } a_0 a_1 \neq 0, \\ &= 0, \qquad \text{if} \quad -\infty < \alpha < 0 \text{ and } a_0 a_1 = 0, \\ &= a_0^{1-\theta} a_1^\theta, \qquad \text{if} \quad \alpha = 0, \\ &= \max(a_0, a_1), \qquad \text{if} \quad \alpha = +\infty, \\ &= \min(a_0, a_1), \qquad \text{if} \quad \alpha = -\infty. \end{aligned} \tag{1.5}$$

We shall present two simple and direct proofs of Theorem 2 following the essence of the original proof of Theorem 1 and the proof of the generalized version of Theorem 1 as given by Hadwiger and Ohman.

A particular case of Theorem 2, useful for multivariate statistical theory, is given below.

THEOREM 3. *Let g be a probability density function on R^n such that for $0 < \theta < 1$*

$$g(x) \geqslant M_\alpha [g(x_0), g(x_1); \theta], \tag{1.6}$$

whenever $x = (1-\theta)\, x_0 + \theta x_1$ and x_0, x_1 are in the support S of g; $-1/n \leqslant \alpha \leqslant +\infty$. Then for any two non-empty measurable set A_0 and A_1 in R^n

$$\int_{(1-\theta)A_0 + \theta A_1} g(x)\, dx \geqslant M_{\alpha_n^*} \left[\int_{A_0} g(x)\, dx, \int_{A_1} g(x)\, dx; \theta \right], \tag{1.7}$$

where α_n^* is given by (1.4), if $-1/n \leqslant \alpha \leqslant 0$, or $0 < \alpha \leqslant +\infty$ and either both $A_0 \cap S$ and $A_1 \cap S$ are non-empty or both are empty.

A non-negative function g satisfying (1.6) for all $\theta (0 < \theta < 1)$ was termed as α-unimodal function by the present author in a previous paper [14]. It may be noted that $(-\infty)$-unimodal functions are precisely the unimodal functions as defined by Anderson [1], and 0-unimodal functions are simply log-concave functions.

Proofs of Theorem 2 and 3 will be given in Section 2. The relevant references, the historical background and further developments will be presented in Section 3. References to some important statistical applications are given. Section 4 gives a review of different concepts of a multivariate unimodal density. In the following, by measurability we mean Borel measurability unless it is specified otherwise.

2. Proofs of Theorems 2 and 3

Proof I of Theorem 2.

Step A. Assume $n = 1$.

(A1) *Basic Lemma* 1. Let a_0, a_1, b_0, b_1 be non-negative numbers. Then for $-1 \leqslant \alpha \leqslant +\infty$

$$M_\alpha(a_0, a_1; \theta)\, M_1(b_0, b_1; \theta) \geqslant M_{\alpha_1^*}(a_0 b_0, a_1 b_1; \theta), \tag{2.1}$$

where α_1^* is given by (1.4).

Proof. The cases $-1 < \alpha < 0$, $0 < \alpha < +\infty$ follow from the general form of Holder's inequality [26, p. 24]. The case $\alpha = 0$ follows from the AM–GM inequality. The result can be easily verified for $\alpha = -1$ and $\alpha = +\infty$.

(A2) Assume that f_i's and S_i's are bounded. First consider the case when $\int_{-\infty}^{\infty} f_0(x)\, dx \int_{-\infty}^{\infty} f_1(x)\, dx = 0$, and $0 < \alpha \leqslant \infty$. Suppose, in particular, $\int_{-\infty}^{\infty} f_0(x)\, dx = 0$. Let $x_0 \in S_0$. Then

$$\begin{aligned}\int_{(1-\theta)S_0+\theta S_1} f(x)\, dx &\geqslant \int_{(1-\theta)x_0+\theta S_1} f(x)\, dx = \int_{S_1} f((1-\theta)\, x_0 + \theta x_1)\, \theta\, dx_1 \\ &\geqslant \int_{S_1} \theta^{1/\alpha} f_1(x_1)\, \theta\, dx_1 \\ &= M_{\alpha_1^*}\left(0, \int_{-\infty}^{\infty} f_1(x)\, dx; \theta\right).\end{aligned}$$

Hence, it is sufficient to assume that

$$\int_{-\infty}^{\infty} f_0(x)\,dx \cdot \int_{-\infty}^{\infty} f_1(x) \neq 0. \tag{2.2}$$

Our proof now uses the well-known *Brunn–Minkowski–Schmidt mapping* (see [4]). For $\eta \in (0, 1)$ define $x_i(\eta)$ by

$$x_i(\eta) = \inf \left\{ t\colon \int_{-\infty}^{t} f_i(x)\,dx \geqslant \eta \int_{-\infty}^{\infty} f_i(x)\,dx \right\}, \qquad (i = 0, 1). \tag{2.3}$$

Let

$$m_i = \int_{-\infty}^{\infty} f_i(x)\,dx, \tag{2.4}$$

$$A_i = \{x_i(\eta)\colon \eta \in (0, 1)\}, \tag{2.5}$$

$$A = \{x(\eta) \equiv (1-\theta)\,x_0(\eta) + \theta x_1(\eta)\colon \eta \in (0, 1)\}. \tag{2.6}$$

Note that $x_i(\eta)$ is strictly increasing in η but it may be discontinuous. The set A_i can be expressed as a countable union of disjoint (bounded) intervals such that inside each such interval

$$f_i(x) \neq 0, \qquad dx_i(\eta)/d\eta = m_i/f_i(x_i(\eta)), \qquad \text{a.e.}$$

(see Natanson [39, Vol. I, p. 253]). Now it can be seen that A can also be expressed as a countable union of disjoint (bounded) intervals such that inside each such interval

$$\begin{gathered} f_0(x)\,f_1(x) \neq 0, \\ \frac{dx(\eta)}{d\eta} = \frac{m_0(1-\theta)}{f_0(x_0(\eta))} + \frac{m_1\theta}{f_1(x_1(\eta))}, \end{gathered} \tag{2.7}$$

a.e. Let A^* be the set obtained from A after excluding from it the above null sets. Clearly $(1-\theta)\,S_0 + \theta S_1 \supset A^*$. Moreover, note that the set of $\eta \in (0, 1)$ for which $x(\eta) \in A^*$ differs from $(0, 1)$ by a null set. Hence

$$\begin{aligned} \int_{(1-\theta)S_0+\theta S_1} f(x)\,dx &\geqslant \int_{A^*} f(x)\,dx \\ &= \int_0^1 f(x(\eta)) \left[\frac{m_0(1-\theta)}{f_0(x_0(\eta))} + \frac{m_1\theta}{f_1(x_1(\eta))} \right] d\eta \end{aligned}$$

$$\geqslant \int_0^1 M_\alpha[f_0(x_0(\eta)), f_1(x_1(\eta)); \theta]$$
$$\cdot M_1(m_0/f_0(x_0(\eta)), m_1/f_1(x_1(\eta)); \theta]\, d\eta$$

$$\geqslant \int_0^1 M_{\alpha_1^*}[m_0, m_1; \theta]\, d\eta$$

$$= M_{\alpha_1^*}(m_0, m_1; \theta),$$

by the basic lemma.

(A3) Supose f_i's are unbounded. Define

$$\begin{aligned} f_{ik}(x) &= f_i(x), && \text{if } f_i(x) \leqslant k, \\ &= k, && f_i(x) > k. \end{aligned} \tag{2.8}$$

Then $f_{ik}(x) \uparrow f_i(x)$ as $k \to \infty$. The inequality (1.3) holds with f_i replaced by f_{ik} $(i = 0, 1)$. An application of the monotone convergence theorem yields the result.

Suppose S_i's are unbounded. Define

$$\begin{aligned} f_{ik}(x) &= f_i(x), && \text{if } |x| \leqslant k, \\ &= 0, && \text{otherwise} \end{aligned} \tag{2.9}$$

Then the support S_{ik} of f_{ik} is $S_i \cap [-k, k]$ which is non-empty for all sufficiently large k. Here again $f_{ik}(x) \uparrow f_i(x)$. Note that $(1-\theta) S_0 + S_1 \supset (1-\theta) S_{0k} + \theta S_{1k}$ for all sufficiently large k. Now (A2) and the monotone convergence theorem yield the result.

Step B. $n \geqslant 1$. Proof by induction on n. Write the first $n-1$ coordinates of $x \in R^n$ as y and the last coordinate of x as z. Let

$$S_i^* = \{z: (y, z) \in S_i \text{ for some } y \in R^{n-1}\}. \tag{2.10}$$

For fixed $z_i \in S_i^*$ and $z = (1-\theta) z_0 + \theta z_1$ write

$$g_i(y) = f_i(y, z_i), \qquad g(y) = f(y, z). \tag{2.11}$$

Let $S_i(z_i)$ be the z_i-section of S_i, i.e.,

$$S_i(z_i) = \{y \in R^{n-1}: (y, z_i) \in S_i\}. \tag{2.12}$$

Clearly $S_i(z_i)$ is non-empty and measurable [25]. Then

$$g(y) \geqslant M_\alpha[g_0(y_0), g_1(y_1); \theta], \tag{2.13}$$

for $y = (1-\theta)\, y_0 + \theta y_1$; $y_i \in S_i(z_i)$; $i = 0, 1$. By the induction hypothesis

$$\int_{(1-\theta) S_0(z_0) + \theta S_1(z_1)} g(y)\, dy \geqslant M_{\alpha^*_{n-1}} \left[\int_{R^{n-1}} g_0(y)\, dy, \int_{R^{n-1}} g_1(y)\, dy; \theta \right].$$

Let

$$h_i(z_i) = \int_{R^{n-1}} g_i(y)\, dy, \qquad (i = 0, 1) \tag{2.14}$$

$$h(z) = \int_{(1-\theta) S_0(z_0) + \theta S_1(z_1)} g(y)\, dy. \tag{2.15}$$

Clearly h_i's and h are measurable [25]. Now note that

$$h(z) \geqslant M_\beta [h_0(z_0), h_1(z_1); \theta],$$

whenever $z = (1-\theta)\, z_0 + \theta z_1$, $z_i \in S^*_1$ $(i = 0, 1)$, $\beta = \alpha^*_{n-1}$. Note that $\beta^*_1 = \alpha^*_n$, $-1 \leqslant \beta \leqslant +\infty$. Clearly, by Fubini's theorem

$$\int_{R^n} f_i(x)\, dx = \int_{S^*_i} h_i(z)\, dz.$$

The support of h is a subset of S^*_i. Moreover, $S \equiv (1-\theta)\, S_0 + \theta S_1$

$$\begin{aligned} \int_{(1-\theta) S_0 + \theta S_1} f(x)\, dx &= \int_{(1-\theta) S^*_0 + \theta S^*_1} \left[\int_{S(z)} f(y, z)\, dy \right] dz \\ &\geqslant \int_{(1-\theta) S^*_0 + \theta S^*_1} \left[\int_{(1-\theta) S_0(z_0) + \theta S_1(z_1)} g(y)\, dy \right] dz \\ &= \int_{(1-\theta) S^*_0 + \theta S^*_1} h(z)\, dz. \end{aligned}$$

It follows from step A that

$$\int_{(1-\theta) S^*_0 + \theta S^*_1} h(z)\, dz \geqslant M_{\beta^*_1} \left[\int_{S^*_0} h_0(z)\, dz, \int_{S^*_1} h_1(z)\, dz; \theta \right].$$

The result now easily follows.

Proof II of Theorem 2.

We start with the assumption made in the step (A2) given above. Excluding the trivial cases we may assume $m_i > 0$ $(i = 0, 1)$. We shall now proceed in several steps.

(a) First assume that f_i is a uniform interval function, i.e.,

$$f_i(x) = c_i \chi(x; I_i), \tag{2.16}$$

where $\chi(\,\cdot\,; I_i)$ denotes the characteristic function of the (bounded) interval I_i, and $c_i > 0$. Then

$$\begin{aligned}\int_{(1-\theta)I_0+\theta I_1} f(x)\,dx &\geqslant M_\alpha[c_0, c_1; \theta]\,\mu[(1-\theta)\,I_0 + \theta I_1]\\ &= M_\alpha[c_0, c_1; \theta][(1-\theta)\,\mu(I_0) + \theta\mu(I_1)]\\ &\geqslant M_{\alpha_1}[c_0\mu(I_0), c_1\mu(I_1); \theta]\end{aligned}$$

by the basic lemma; μ denotes te Lebesgue measure on R^1.

(b) Next assume that f_i is a step function, i.e.,

$$f_i(x) = \sum_{j=1}^{p_i} c_{ij}\chi(x; I_{ij}), \qquad (i = 0, 1), \tag{2.17}$$

where $c_{ij} > 0$ and $I_{ij}(j = 1,..., p_i)$ are pairwise disjoint (bounded) intervals. We shall now employ a technique known as *Hadwiger–Ohman cut* [24]. Let

$$I_i = \bigcup_{j=1}^{p_i} I_{ij}, \qquad (i = 0, 1). \tag{2.18}$$

Let b_0 be a real number such that the number of I_{0j}'s to the left of b_0 and the number of I_{0j}'s to the right of b_0 are both positive, the total number being p_0. Write

$$\begin{aligned}f_0(x) &= f_0(x)\,\chi(x; x \leqslant b_0) + f_0(x)\,\chi(x; x > b_0)\\ &\equiv f_{01}(x) + f_{02}(x).\end{aligned} \tag{2.19}$$

Let b_1 be a real number such that

$$\int_{-\infty}^{b_1} f_1(x)\,dx/m_1 = \int_{-\infty}^{b_0} f_0(x)\,dx/m_0. \tag{2.20}$$

Such a b_1 can be found. Write

$$\begin{aligned}f_1(x) &= f_1(x)\,\chi(x; x \leqslant b_1) + f_1(x)\,\chi(x; x > b_1)\\ &\equiv f_{11}(x) + f_{12}(x).\end{aligned} \tag{2.21}$$

Then f_{ij} $(j = 1, 2)$ can be expressed as a step function with the number of disjoint intervals defining f_{ij} less than or equal to p_i. We shall now prove the result by induction on $p_0 + p_1$.

Let I_i^* and I_i^{**} be the supports of f_{i1} and f_{i2}, respectively. By the induction hypothesis

$$\begin{aligned}\int_{(1-\theta)I_0^*+\theta I_1^*} f(x)\,dx + \int_{(1-\theta)I_0^{**}+\theta I_1^{**}} f(x)\,dx \\ \geqslant M_{\alpha_1^*}\left[\int_{-\infty}^{\infty} f_{01}(x)\,dx, \int_{-\infty}^{\infty} f_{11}(x)\,dx; \theta\right] \\ + M_{\alpha_1^*}\left[\int_{-\infty}^{\infty} f_{02}(x)\,dx, \int_{-\infty}^{\infty} f_{12}(x)\,dx; \theta\right] \\ = M_{\alpha_1^*}\left[\int_{-\infty}^{\infty} f_0(x)\,dx, \int_{-\infty}^{\infty} f_1(x)\,dx; \theta\right]\end{aligned}$$

using (2.20). Note that $(1-\theta)\,I_0^* + \theta I_1^*$ and $(1-\theta)\,I_0^{**} + \theta I_1^{**}$ are disjoint and their union is included in$(1-\theta)\,I_0 + I_1$. The desired result now easily follows.

(c) Assume now

$$f_i(x) = \sum_{j=1}^{p_i} c_{ij}\chi(x; B_{ij}), \qquad (i = 0, 1), \tag{2.22}$$

where $c_{ij} > 0$ and B_{ij} $(j = 1,..., p_i)$ are pairwise disjoint compact sets in R^1. Without loss of generality, we may assume $\mu(\beta_{ij}) > 0$. It is possible to find a sequence $\{I_{ij}^{(k)}\}$ such that each $I_{ij}^{(k)}$ is a finite union of disjoint (bounded) intervals and $I_{ij}^{(k)} \downarrow B_{ij}$ as $k \to \infty$; moreover $I_{ij}^{(k)}$ and $I_{ij'}^{(k)}$ are disjoint. Define

$$f_i^{(k)}(x) = \sum_{j=1}^{p_i} c_{ij}\chi(x; I_{ij}^{(k)}), \qquad (i = 0, 1) \tag{2.23}$$

and

$$\begin{aligned}f^{(k)}(x) = \max\{M_\alpha(c_{0j}, c_{1j'}; \theta); x = (1-\theta)\,x_0 + \theta x_1, \\ x_0 \in I_{0j}^{(k)}, x_1 \in I_{1j'}^{(k)} \text{ for some } j, j'\}.\end{aligned} \tag{2.24}$$

Let

$$I_i^{(k)} = \bigcup_j I_{ij}^{(k)}. \tag{2.25}$$

Then, by the result in (b)

$$\int_{(1-\theta)I_0^{(k)}+\theta I_1^{(k)}} f^{(k)}(x)\,dx \geqslant M_{\alpha_1^*}\left(\int_{-\infty}^{\infty} f_0^{(k)}(x)\,dx, \int_{-\infty}^{\infty} f_1^{(k)}(x)\,dx; \theta\right). \tag{2.26}$$

Since

$$f_i^{(k)}(x) \downarrow f_i(x), \int_{-\infty}^{\infty} f_i^{(k)}(x)\, dx \to \int_{-\infty}^{\infty} f_i(x)\, dx \qquad (i=0, 1).$$

Note that

$$I_\theta^{(k)} \equiv (1-\theta)\, I_0^{(k)} + \theta I_1^{(k)} = (1-\theta)\left(\bigcup_j I_{0j}^{(k)}\right) + \theta\left(\bigcup_j I_{1j}^{(k)}\right) \tag{2.27}$$

converges to ($I_{ij}^{(k)}$ can be suitably so chosen)

$$(1-\theta)\left(\bigcup_j B_{0j}\right) + \theta\left(\bigcup_j B_{1j}\right) \equiv (1-\theta)\, B_0 + \theta B_1 \equiv B_\theta. \tag{2.28}$$

Let

$$f^*(x) = \max\{M_\alpha(c_{0j}, c_{1j'}; \theta); x = (1-\theta)\, x_0 + \theta x_1,$$
$$x_0 \in B_{0j},\ x_1 \in B_{1j'} \text{ for some } j \text{ and } j'\}. \tag{2.29}$$

Now

$$\int_{I_\theta^{(k)}} f^{(k)}(x)\, dx = \int_{B_\theta} f^{(k)}(x)\, dx + \int_{I_\theta^{(k)} - B_\theta} f^{(k)}(x)\, dx, \tag{2.30}$$

which converges to $\int_{B_\theta} f^*(x)\, dx$, since $f^{(k)}(x) \downarrow f^*(x)$ for $x \in B_\theta$, $f^{(k)}(x)$ is bounded and $I_\theta^{(k)} \downarrow B_\theta$. The result now follows from the fact that

$$f(x) \geqslant f^*(x) \tag{2.31}$$

for $x \in B_\theta$.

(d) Assume now f_i is a simple function, i.e.,

$$f_i(x) = \sum_{j=1}^{p_i} c_{ij} \chi(x; A_{ij}), \qquad (i=0, 1), \tag{2.32}$$

where $c_{ij} > 0$ and A_{ij} $(j = 1, \ldots, p_i)$ are pairwise disjoint (bounded) measurable sets in R^1; without loss of generality, we may assume that $0 < \mu(A_{ij}) < \infty$. Given A_{ij} there exists a sequence of compact sets $B_{ij}^{(k)}$ such that $B_{ij}^{(k)} \subset A_{ij}$ and $\mu(B_{ij}^{(k)}) \uparrow \mu(A_{ij})$ as $k \to \infty$. Define

$$f_i^{(k)}(x) = \sum_{j=1}^{p_i} c_{ij} \chi(x; B_{ij}^{(k)}), \qquad (i=0, 1). \tag{2.33}$$

Note that $f_i^{(k)}(x) \leqslant f_i(x)$ and $f_i^{(k)}(x) \to f_i(x)$ in μ-measure. Thus by the dominated convergence theorem

$$\int_{-\infty}^{\infty} f_i^{(k)}(x)\,dx \to \int_{-\infty}^{\infty} f_i(x)\,dx, \qquad (i=0,1). \tag{2.34}$$

The desired result now easily follows.

(e) General case. Given f_i there exists an increasing sequence $\{f_i^{(k)}\}$ of non-negative simple functions such that

$$f_i^{(k)}(x) \to f(x), \qquad \int_{-\infty}^{\infty} f_i^{(k)}(x)\,dx \to \int_{-\infty}^{\infty} f_i(x)\,dx. \tag{2.35}$$

Let $S_i^{(k)}$ be the support of $f_i^{(k)}$. Then $S_i^{(k)} \subset S_i$. The result now follows from (d).

(f) After proving the case in (A2) we can use the remaining steps in Proof I to complete the proof. Alternatively, the above proof can be easily modified to cover the general case $n \geqslant 1$. The only crucial change occurs in step (a). For this, consider I_i as the cartesian product of n intervals of respective lengths $l_{i1},..., l_{in}$. Then

$$\begin{aligned}
\int_{(1-\theta)I_0+\theta I_1} f(x)\,dx &\geqslant M_\alpha[c_0, c_1; \theta]\,\mu_n[(1-\theta)\,I_0 + \theta I_1] \\
&= M_\alpha[c_0, c_1; \theta] \prod_{j=1}^{n} [(1-\theta)\,l_{0j} + \theta l_{ij}] \\
&= M_\alpha[c_0, c_1; \theta] \prod_{j=1}^{n} M_1(l_{0j}, l_{1j}; \theta) \\
&\geqslant M_{\alpha_n^*}\left(c_0 \prod_{j=1}^{n} l_{0j}, c_1 \prod_{j=1}^{n} l_{1j}; \theta\right)
\end{aligned} \tag{2.36}$$

by applying the basic lemma successively.

Proof of Theorem 3. Suppose $A_0 \cap S$ and $A_1 \cap S$ are both non-empty. Define

$$f_i(x) = g(x)\,\chi(x; A_i). \tag{2.37}$$

Then Theorem 2 yields

$$\begin{aligned}
M_{\alpha_n^*}\left(\int_{R^n} f_0(x)\,dx, \int_{R^n} f_1(x)\,dx; \theta\right) &\leqslant \int_{(1-\theta)A_0\cap S+\theta A_1\cap S} g(x)\,dx \\
&\leqslant \int_{(1-\theta)A_0+\theta A_1} g(x)\,dx.
\end{aligned} \tag{2.38}$$

Clearly

$$\int_{R^n} f_i(x)\,dx = \int_{A_i} g(x)\,dx. \tag{2.39}$$

The Theorem follows easily if $\int_{A_0} g(x)\,dx \cdot \int_{A_1} g(x)\,dx = 0$, $\alpha \leqslant 0$, or $\int_{A_0} g(x)\,dx = \int_{A_1} g(x)\,dx = 0$, $\alpha > 0$.

Remarks. (1) One may raise the question whether $(1-\theta)\,S_0 + \theta S_1$ in Theorem 2 or $(1-\theta)\,A_0 + \theta A_1$ in Theorem 3 are measurable. It is known [21] that the Minkowski sum of two Borel sets in R^n may not be Borel; however, it is analytic and hence it is Lebesgue measurable [39, Vol. II, p. 250]. If we want to deal with Lebesgue-measurable functions and sets the left-hand sides of (1.3) and (1.7) should be replaced by the respective lower integrals (i.e., the inner measure induced by the respective functions). To avoid the measurability problem Henstock and Macbeath [26] considered S_0 and S_1 to be $\mathscr{F}_\sigma$ sets so that $S_0 + S_1$ is also an $\mathscr{F}_\sigma$ set; in this connection see Hadwiger and Ohman [24] and Dinghas [18].

(2) It is possible to formulate Theorem 2 in the following way. One may replace (1.2) by the same condition with $x = (1-\theta)\,x_0 + \theta x_1$, $x_0 \in A_0$, $x_1 \in A_1$, when A_0 and A_1 are non-empty measurable sets in R^n. In that case we shall assume $\int_{A_i} f_i(x)dx < \infty$ $(i = 0, 1)$. Then (1.3) would be replaced by the following:

$$\int_{(1-\theta)A_0+\theta A_1} f(x)\,dx \geqslant M_{\alpha_n^*}\left[\int_{A_0} f_0(x), \int_{A_1} f_1(x)\,dx; \theta\right]. \tag{2.40}$$

If both $A_0 \cap S_0$ and $A_1 \cap S_1$ are non-empty then Theorem 2 yields

$$\int_{(1-\theta)A_0\cap S_0+\theta A_1\cap S_1} f(x)\,dx \geqslant M_{\alpha_n^*}\left[\int_{A_0\cap S_0} f_0(x)\,dx, \int_{A_1\cap S_1} f_1(x)\,dx; \theta\right], \tag{2.41}$$

which is stronger than (2.40). If both $A_0 \cap S_0$ and $A_1 \cap S_1$ are empty, then (2.41) follows trivially. On the other hand, if only $A_0 \cap S_0$ is empty and $0 < \alpha \leqslant \infty$ (the result follows trivially if $\alpha \leqslant 0$) we take $x_0 \in A_0$ and use the following:

$$\begin{aligned}\int_{(1-\theta)A_0+\theta A_1} f(x)\,dx &\geqslant \int_{(1-\theta)x_0+\theta A_1} f(x)\,dx = \int_{A_1} \theta^n f((1-\theta)\,x_0 + \theta x_1)\,dx_1 \\ &\geqslant \int_{A_1} \theta^n \theta^{1/\alpha} f_1(x_1)\,dx_1 \qquad (2.42) \\ &= M_{\alpha_n^*}\left(0, \int_{A_1} f_1(x)\,dx; \theta\right).\end{aligned}$$

(3) As in Remark 2, one may also reformulate Theorem 3 by requiring (1.6) to hold for $x=(1-\theta)x_0+\theta x_1$ with $x_0 \in A_0$, $x_1 \in A_1$. However, if only one of $A_i \cap S$ is empty (1.6) may not hold with many such g's although (1.7) still holds.

(4) We could have also formulated Theorem 3 without requiring g to be a probability density function. In that case we would assume $\int_{A_i} g(x)\,dx < \infty$ $(i=0, 1)$, where g is a non-negative measurable function.

Now we shall show that Theorem 3 can be proved directly, simply by using Brunn–Minkowski–Lusternik inequality. First we shall prove the following lemma which is stronger than Theorem 2 when f_i's are bounded, and $n=1$.

Lemma 2. *Let f_0, f_1 be non-negative, bounded, measurable functions on R^1. Suppose f_i's are integrable with respect to μ_1 (Lebesgue measure on R^1). Let f be a non-negative, measurable function on R^1 such that*

$$f(x) \geqslant M_\alpha[f_0(x_0), f_1(x_1); \theta],$$

whenever $x=(1-\theta)x_0+\theta x_1, x_i \in S_i$ $(i=0, 1)$, where S_i is the support of f_i; S_i's are assumed to be non-empty. Then

$$\int_{(1-\theta)S_0+\theta S_1} f(x)\,dx \geqslant M_\alpha(c_0, c_1; \theta)\, M_1\left[c_0^{-1}\int_{-\infty}^{\infty} f_0(x)\,dx, c_1^{-1}\int_{-\infty}^{\infty} f_1(x)\,dx; \theta\right],$$

where c_i is the supremum of f_i.

Proof. Define

$$E_i=\{x^*=(x, z) \in R^2: f_i(x) > zc_i, z>0, x \in S_i\}, \qquad (i=0, 1)$$

$$E=\{x^*=(x, z) \in R^2: f(x) > zM_\alpha(c_0, c_1; \theta); z>0, x \in (1-\theta)S_0+\theta S_1\}.$$

Let $E_i(z)$ and $E(z)$ be the z-sections of E_i and E, respectively. For $0<z<1$ both $E_0(z)$ and $E_1(z)$ are non-empty, and

$$E(z) \supset (1-\theta)E_0(z)+\theta E_1(z).$$

Moreover,

$$\int_{-\infty}^{\infty} f_i(x)\,dx=\int_0^1 \mu_1(E_i(z))\,dz\, c_i,$$

$$\int_{(1-\theta)S_0+\theta S_1} f(x)\,dx \geqslant \int_0^1 \mu_1(E(z))\,dz \cdot M_\alpha(c_0, c_1; \theta).$$

By the one-dimensional Brunn–Minkowski–Lusternik inequality

$$\mu_1(E(z)) \geqslant (1-\theta)\,\mu_1(E_0(z)) + \theta\mu_1(E_1(z)),$$

for $0 < z < 1$. The result now follows easily.

Proof III of Theorem 2.

In view of (A3) and Step B of Proof I it is sufficient to prove the theorem when the f_i's are bounded and $n = 1$. From Lemma 2 we get

$$\begin{aligned}\int_{(1-\theta)S_0+\theta S_1} & f(x)\,dx \\ & \geqslant M_\alpha(c_0, c_1; \theta)\, M_1\left[c_0^{-1}\int_{-\infty}^{\infty} f_0(x)\,dx, c_1^{-1}\int_{-\infty}^{\infty} f_1(x)\,dx; \theta\right] \\ & \geqslant M_{\alpha_1^*}\left[\int_{-\infty}^{\infty} f_0(x)\,dx, \int_{-\infty}^{\infty} f_1(x)\,dx; \theta\right],\end{aligned}$$

using the basic lemma.

3. Historical Developments

Theorem 2 is essentially contained in the original proof of Brunn–Minkowski inequality in the following form: $\alpha = 1/(m-1)$, $n = 1$; f, f_0 and f_1 are non-negative bounded continuous functions; S_0 and S_1 are bounded intervals.

Later the following special case of (essentially) Theorem 2 was proved by Henstock and Macbeath [27]: $0 < \alpha < \infty$; $n = 1$; f, f_0, f_1 are taken as non-negative, bounded, measurable functions, where

$$f(x) = \sup_{x=(1-\theta)x_0+\theta x_1} M_\alpha^*[f_0(x_0), f_1(x_1); \theta], \tag{3.1}$$

$$\begin{aligned} M_\alpha^*(a_0, a_1; \theta) &= M_\alpha(a_0, a_1, \theta), \qquad \text{if} \quad a_0 a_1 \neq 0 \\ &= 0, \qquad \text{otherwise.}\end{aligned} \tag{3.2}$$

The final result is also given in terms of M_α^* instead of M_α. However, throughout their development both $1-\theta$ and θ were replaced by 1 in defining f, as well as, in the final result. This result was extended by Dinghas [18] to the case $n \geqslant 1$ in the direction discussed in Remark 2. Dinghas introduced a generalized integral (following Saks) and considered the case when f, f_0 and f_1, A_0 and A_1 are not necessarily measurable. In all the above results a special case of the basic lemma and Brunn–Minkowski–Schmidt mapping were used.

Theorem 2 for $\alpha = 0$ and $n = 1$ was proved by Prekopa [45] when $\theta = 1/2$, and by Leindler [33] when $0 < \theta < 1$. Later, Prekopa [46] proved Theorem 2 for $\alpha = 0$, $n \geqslant 1$ using induction on n. In all these results

$$f(x) = \sup_{x=(1-\theta)x_0+\theta x_1} M_0[f_0(x_0), f_1(x_1); \theta]. \tag{3.3}$$

Subsequently Prekopa [47] derived Theorem 3 for $\alpha = 0$, and derived conditions for which the inequality is strict. However, the proofs of Prekopa and Leindler are quire obscure and somewhat incomplete.

Theorem 2 in a more general form was proved by Borell [5] in 1975 following the techniques of Hadwiger and Ohman [24] and Dinghas [18]. However, Borell's proof is unnecessarily lengthy and not easily comprehensible.

A special case of Theorem 2 (and of Theorem 3) can be proved by using the following weak (although apparently simple) method. Define

$$B_i = \{x^* = (x, z) \in R^{n+1}: f_i(x) > q_\alpha(z), x \in S_i\}, \qquad (i = 0, 1) \tag{3.4}$$

$$B = \{x^* = (x, z) \in R^{n+1}: f(x) > q_\alpha(z), x \in (1-\theta) S_0 + \theta S_1\}, \tag{3.5}$$

where

$$\begin{aligned} q_\alpha(z) &= z^{1/\alpha}, && \text{if} \quad \alpha \neq 0, \alpha < \infty, \text{and } z > 0 \\ &= \exp(-z), && \alpha = 0. \end{aligned} \tag{3.6}$$

B_i is not defined for $z \leqslant 0$ when $\alpha \neq 0$. Let $B_i(z)$ and $B(z)$ be the z-sections of B_i and B, respectively. Then

$$\int_{R^n} f_i(x)\, dx = \int_{-\infty}^{\infty} \mu_n(B_i(z))\, h_\alpha(z)\, dz, \tag{3.7}$$

where

$$\begin{aligned} h_\alpha(x) &= |\alpha|^{-1} z^{(1/\alpha)-1} \chi(z: z > 0), && \text{if} \quad \alpha \neq 0, \quad \alpha < \infty \\ &= \exp(-z), && \text{if} \quad \alpha = 0. \end{aligned} \tag{3.8}$$

Let I_i be the support of $\mu_n(B_i(z))$. Then for $z_0 \in I_0$, $z_1 \in I_1$

$$B((1-\theta) z_0 + \theta z_1) \supset (1-\theta) B_0(z_0) + \theta B_1(z_1), \tag{3.9}$$

and by Brunn–Minkowski–Lusternik inequality we get

$$\mu_n[B((1-\theta) z_0 + \theta z_1)] \geqslant M_{1/n}[\mu_n(B_0(z_0)), \mu_n(B_1(z_1)); \theta]. \tag{3.10}$$

Note now

$$\int_{(1-\theta)S_0+\theta S_1} f(x)\,dx \geqslant \int_{(1-\theta)I_0+\theta I_1} \mu_n(B(z))\,h_\alpha(z)\,dz. \qquad (3.11)$$

It follows from the general form of Holder's inequality [26, p. 24]

$$\mu_n(B(z))\,h_\alpha(z) \geqslant M_\beta(\mu_n(B_0(z_0))\,h_\alpha(z_0), \mu_n(B_1(z_1))\,h_\alpha(z_1); \theta), \qquad (3.12)$$

where $z=(1-\theta)z_0+z_1$, $z_i \in I_i$ $(i=0,1)$ and $\beta=\gamma_n^*$, $\gamma=\alpha/(1-\alpha)$, provided $-1/n \leqslant \gamma < \infty$. Note that $-1/n \leqslant \gamma < \infty$ is equivalent to $-1/(n+1) \leqslant \alpha < 1$. When $\alpha=1$, (3.12) is the same as (3.10) with $\beta=1/n$.

Suppose now Theorem 2 is true for $n=1$. Then

$$\int_{(1-\theta)I_0+\theta I_1} \mu_n(B(z))\,h_\alpha(z)\,dz$$

$$\geqslant M_{\beta_1^*}\left(\int_{-\infty}^{\infty} \mu_n(B_0(z))\,h_\alpha(z)\,dz, \int_{-\infty}^{\infty} \mu_n(B_1(z))\,h_\alpha(z)\,dz; \theta\right),$$

where $\beta_1^*=\alpha_n^*$, provided $-1 \leqslant \beta$ (i.e., $-1/(n+1) \leqslant \gamma \leqslant \infty$, which is equivalent to $-1/n \leqslant \alpha \leqslant 1$). So the problem now reduces to proving Theorem 2 for $n=1$; even then Theorem 2 will be proved for $n \geqslant 1$ and only for $-1/n \leqslant \alpha \leqslant 1$.

The above idea of using epigraph is not new. It can be found in Bonneson [3], Henstock and Macbeath [27]; Das Gupta [14] also mentioned this reduction. Rinott [48] in 1976 used the above idea to prove Theorem 3 for $-1/n \leqslant \alpha < 1$ and Theorem 2 for $1/n \leqslant \alpha \leqslant 0$. Essentially Rinott proved Theorem 2 for some special α and $n=1$ using Brunn–Minkowski–Schmidt mapping; however, his proof is not rigorous. It is obvious that the proof by induction on n is much easier and *does not* restrict α to $-1/n \leqslant \alpha \leqslant 1$.

Proof III of Theorem 2 is most elegant if one is allowed to use one-dimensional Brunn–Minkowski–Lusternik inequality. This proof using Lemma 2 was given by Bonneson [3] for convex sets. Later, Henstock and Macbeath [27] extended Bonneson's result to the special case of Theorem 2 for $0<\alpha<\infty$ after proving Lemma 2. (However, Henstock and Macbeath [27] replaced both θ and $1-\theta$ by 1 and M_α by M_α^*.) In 1975, Brascamp and Lieb [7] used Lemma 2 and the basic lemma to furnish Proof III of Theorem 2; Proof III is really trivial once these two lemmas are known. Brascamp and Lieb [7] considered

$$f(x) = \operatorname*{ess\,sup}_{x=(1-\theta)x_0+x_1} M_\alpha^*[f_0(x_0), f_1(x_1); \theta],$$

and instead of the Minkowski sum of two sets A_0 and A_1 they considered

$$\operatorname{ess}\{(1-\theta)A_0+\theta A_0\}$$
$$\equiv \{x\colon (x-(1-\theta)A_0)\cap \theta A_1 \text{ has } + \text{ve } \mu_n\text{-measure}\}.$$

It was shown that for non-negative measurable f_0 and f_1, f is lower semi-continuous; for measurable A_0 and A_1, $\operatorname{ess}(A_0+A_1)$ is open.

It is easy to see that Theorem 2 for $\alpha=0$ implies Brunn–Minkowski–Lusternik inequality [7]. Hence in order to get all the above results it is sufficient either to prove Theorem 2 for $\alpha=0$ and $n=1$, or the one-dimensional version of Brunn–Minkowski–Lusternik inequality. All the other results then follow quite easily. The one-dimensional version of B–M–L inequality was simply stated by Lusternik [34]; a rigorous proof for a somewhat stronger results is given in Henstock and Macbeath [27]. Otherwise, proofs I and II can be adopted for this purpose. Brascamp and Lieb [6] presented four proofs leading to Theorem 3 for $\alpha=0$ and $n=1$. However, their proofs either use Brunn–Minkowski–Lusternik inequality or use essentially Brunn–Minkowski–Schmidt mapping (as in A2 in Proof I).

Thus it appears that after the pioneering work of Brunn–Minkowski–Lusternik, the works of Bonneson [3], Henstock and Macbeath [27] and Hadwiger and Ohman [24] are the only important ones. Proofs of the subsequent results are not new, although these results point out some simple but useful extensions. In Section 2 we have presented the important steps with necessary modifications and elaborations.

The conditions for which the inequalities in Theorems 2 and 3 are strict are not stated explicitly in the literature except for the case $\alpha=0$ [46]. However, Proof III along with the work of Henstock and Macbeath [26] would yield the desired conditions.

Th following converse of Theorem 3 was proved by Borell [5].

THEOREM (Borell). (a) *Let Ω be an open convex subset of R^n and let μ be a positive Radon measure in Ω such that*

$$\mu((1-\theta)A_0+\theta A_1)\geqslant \min(\mu(A_0),\mu(A_1))$$

for all semi-open blocks A_0 and A_1 in Ω and all $0\leqslant\theta\leqslant 1$. Then the support S_μ of μ is convex, and if $\dim(S_\mu)=n$ then μ is absolutely continuous with respect to μ_n.

(b) *Let μ be a positive Radon measure in an open convex set $\Omega\subset R^n$ such that for*

$$\mu_*((1-\theta)A_0+\theta A_1)\geqslant M_s[\mu_*(A_0),\mu_*(A_1);\theta]$$

for all non-empty sets A_0 and A_1 in Ω. Let H be the least affine subspace *which contains* S_μ *and* $m = \dim(H)$. *Then* $d\mu = f\, d\mu_n$ *and* f *is* α*-unimodal, where* $s = \alpha_m^*$ *for* $-\infty \leqslant s \leqslant 1/n$ $(m = n$ *if* $s > 0)$ *and* $f = 0$ *for* $s > 1/n$.

A simpler version of part (b) of the above Theorem is proved by Rinott [48]. Borell [5] also proved a similar converse of Theorem 2.

It may be noted that Theorem 2 and 3 do not apply when $\alpha < -1/n$. Although a good many p.d.f.'s satisfy (1.6) for $-1/n \leqslant \alpha$, some general results are sought for unimodal functions (for which $\alpha = -\infty$). With an additional assumption of central (about the origin) symmetry the following use of Brunn–Minkowski inequality by Anderson [1] led to many useful results.

THEOREM (Anderson). *Let* f *be a centrally symmetric, unimodal, non-negative, integrable function on* R^n, *and* C *be a centrally symmetric convex set in* R^n. *Define*

$$h(y) = \int_{R^n} f(x+y)\, \chi(x; C)\, dx.$$

Then h *is centrally symmetric ray-unimodal, i.e.,*

$$h(y) = h(-y), \qquad h(\lambda y) \geqslant h(y)$$

for all $0 \leqslant \lambda \leqslant 1$, *and all* $y \in R^n$.

This result was slightly extended by Sherman [51], (the basic idea in Sherman's work is contained in Fary, I. and Redei, L. (1950). *Math. Ann.* **122** 205–220) and generalized to the case of invariance under a measure-preserving linear group of transformations (instead of central-symmetry) by Mudholkar [38]. For further generalization in terms of marginalization see Das Gupta [13].

Anderson's result follows easily from Brunn–Minkowski inequality when f is the characteristic function of a centrally symmetric convex set. Now to get Anderson's theorem simply note that

$$f(x) = \int_0^\infty \chi(x, z; f(x) \geqslant z)\, dz.$$

By using a similar argument we can say that Anderson's theorem holds when $\chi(x; C)$ is replaced by centrally symmetric, unimodal function g provided the integrals involved are finite. Another extension is given by Das Gupta [13] following the above line of proof.

THEOREM (Das Gupta). *Let $f(x, y)$ be a centrally symmetric unimodal function on $R^n \times R^m$ such that $f(x, y)$ is integrable with respect to μ_n for each fixed y. Then*

$$f_1(y) \equiv \int_{R^n} f(x, y)\, \mu_n(dx)$$

is centrally symmetric ray-unimodal.

Note that f_1, as given above, is also unimodal when $m = 1$. The above theorem in turn leads to the following results:

(a) The convolution of two 0-unimodal densities is 0-unimodal.

(b) A marginal p.d.f. obtained from a 0-unimodal joint p.d.f. is 0-unimodal.

(c) Brunn–Minkowski inequality (i.e., for convex sets).

(d) Theorem 3 for $\alpha = 0$ when A_0 and A_1 are convex.

Note that all the above results follow from Theorem 2; nevertheless they also follow from Das Gupta's Theorem which is a simple extension of Anderson's theorem. The key for these proofs is the following. If g is a 0-unimodal function defined on $R^n \times R^m$ then

$$f(y, v; x, u) \equiv g(x - y, (u - v)/2)\, g(x + y, (u + v)/2)$$

is a centrally symmetric unimodal function in (y, v) for every (x, u). This fact was first noted by Davidovic, Korenbljum and Hacet [14] and later by Brascamp and Lieb [6]. The above fact is used to show

$$h^2(x) \geqslant h(x + y)\, h(x - y),$$

where

$$h(x) = \int_{R^m} g(x, u)\, du.$$

Result (a) is given in [16] (see [28] and [50] for $n = 1$) and Result (b) in [47, 5] and [6].
To prove (c) from (b) simply note that for any two convex sets A_0 and A_1 in R^n the characteristic function of the set

$$D = \{(\theta, x); \theta \in [0, 1], x \in (1 - \theta) A_0 + \theta A_1\}$$

is 0-unimodal (see [6]). Note now (excluding the trivial cases)

$$(1 - \theta) A_0 + \theta A_1 = [(1 - \eta) A_0^* + \eta A_1^*][(1 - \theta)\, \mu_n^{1/n}(A_0) + \theta \mu_n^{1/n}(A_1)],$$

where

$$\eta = \theta\mu_n^{1/n}(A_1)/[(1-\theta)\mu_n^{1/n}(A_0) + \theta\mu_n^{1/n}(A_1)], \qquad A_i^* = A_i/\mu_n^{1/n}(A_i).$$

To prove (d) from (b) consider $g(x)\chi(\theta, x; D)$, where g is a 0-unimodal function. Note that for (a)–(d) we need only Das Gupta's Theorem for $n=1$; this can be proved using the *one-dimensional Brunn–Minkowski inequality for intervals.*

Anderson's Theorem is also used to show that Schur-concavity of p.d.f.'s is closed under convolution [35]. A p.d.f. g on R^n is said to be Schur-concave if $g(y) \geqslant g(x)$ for every x, y such that y is a convex combination of permutations of x. One of the key facts to show this is the following: For a (non-negative) Schur-concave function g on R^n

$$g(u+v, u-v, x_3, \ldots, x_n)$$

is central-symmetric unimodal, as a function of v only. See [20] for an extension of this result.

4. Unimodal Probability Measures

Applications of Brunn–Minkowski inequality to statistical theory were primarily concerned with probability measures which are unimodal in some sense. Several attempts were made to translate the geometric notion of unimodality in R^n into analytic forms.

(a) The earliest attempt was made by Anderson [1] who called a probability distribution in R^n symmetric unimodal (SUM) if it possesses a density f with respect to the Lebesgue measure μ_n such that the sets $\{x: f(x) \geqslant c\}$ for $c \in [0, \infty)$ are convex and symmetric about the origin whenever they are non-empty. Following this Dharmadhikari and Jogdeo [17] called a distribution convex UM about 0 if the sets $\{x: f(x) \geqslant c\}$ for $c \in [0, \infty)$ are convex and contain 0 whenever they are non-empty.

(b) Sherman [51] generalized Anderson's definition by considering f as a member of the closure (with respect to the maximum of the sup-norm and the L_1-norm) of the convex cone generated by the indicator functions of compact, symmetric convex sets in R^n containing 0 in their interiors.

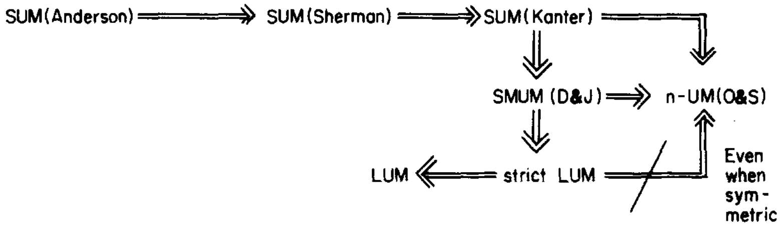
(c) Olshen and Savage [40] defined a r.v. X in R^n to be α-unimodal about 0, if for all real, bounded, non-negative Borel functions g on R^n the function $t^\alpha \mathscr{E}[g(tx)]$ decreases as t increases in $[0, \infty)$. When X has a p.d.f. f with respect to μ_n this definition is equivalent to the requirement that $t^{n-\alpha} f(tx)$ is decreasing for all fixed x as t increases in $[0, \infty)$.

(d) Dharmadhikari and Jogdeo [17] called a r.v. X in R^n linear unimodal (LUM) if for every vector a in R^n the distribution of $a'X$ is unimodal (in the univariate sense). When every such linear combination $a'X$ has a unimodal distribution about 0 the r.v. X is said to be strictly linear unimodal about 0. This definition was also introduced by Ghosh [23].

(e) Dharmadhikari and Jogdeo [17] called a probability measure P on R^n (symmetric) monotone UM (SMUM) if for every convex set C in R^n symmetric about 0 the quantity $P(C+ky)$ is non-increasing in $k \in [0, \infty)$ for every fixed non-zero vector $y \in R^n$.

(f) Kanter [30] defined a probability measure on R^n to be symmetric unimodal if it is a generalized mixture (in the sense of integrating with respect to a probability measure) of all uniform probability measures on symmetric, compact, convex sets in R^n. It essentially gives the closed (in the sense of weak convergence) convex hull generated by such uniform probability measures.

The current status regarding the inter-relationships of these definitions of unimodality in R^n can be described as follows (see [17, 30, 54]):



(⟹ : strict implication)

Although the strict LUM is a natural generalization of the univariate UM, there are examples to indicate that such a distribution may have a "crater." On the other hand, if a p.d.f. in R^n fails to be n-UM then it should not be unimodal in any sense. The problem here is to give an analytic definition of a mode in R^n. In the general case where symmetry is not assumed Kanter's definition (dropping the symmetry part) may be used; the validity of this definition is not yet analysed.

In practice one looks for a definition of unimodality such that the set of all such unimodal distributions is closed under convolution, marginality, product measures, and weak convergence. It is known that Anderson's definition for SUM does not meet any of these requirements, whereas Kanter's definition meets all of them.

It was shown by Lapin (see [31]) and later by Chernin and Ibragimov (see [29]) that all stable densities in R are unimodal. Lapin's proof is known to be false and recently Kanter [31] has indicated that the proof of Chernin and Ibragimov contains an essential gap. Wolfe [56] has shown that every n-dimensional, symmetric distribution function of class L is unimodal in Kanter's sense. It is now known that all L class densities are unimodal.

5. Applications

Anderson's inequality along with its generalizations as derived from Brunn–Minkowski inequality, was used in the literature to obtain many interesting results in multivariate distribution theory and multivariate statistical inference (studies of power functions and confidence regions). See [1, 2, 9–11, 14, 15, 19, 32, 37, 41–43, 52, 53]. For applications in stochastic processes see [1, 7]. For other statistical applications see [44, 49].

Acknowledgments

The author is thankful to Professor W. Sudderth and Professor T. Armstrong for some helpful discussions.

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